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SOLUTION OF THE GENERAL EQUATION OF THE FIFTH DEGREE.

TRANSLATED BY DR. A. B. NELSON, DANVILLE, KENTUCKY.

(*Continued from page* 173, *Vol. III.*)

§. 11.

The function $b = (yv+yz+yu+yt+vz+vu+vt+zu+zt+ut)$ is symmetrically halved, if I separate it into two parts (of which each has 5 terms) whose product $= y^2v^2z^2u^2t^2$.

There are 12 such symmetrical halves, viz.:

I. $(yv+yt+vz+zu+ut) \quad +(yz+yu+vu+vt+zt)$
$(yz+yu+vz+vt+ut) \quad +(yv+yt+vu+zu+zt)$
$(yv+yu+vt+zu+zt) \quad +(yz+yt+vz+vu+ut)$
$(yz+yt+vu+vt+zu) \quad +(yv+yu+vz+zt+ut)$
$(yv+yz+vu+zt+ut) \quad +(yu+yt+vz+vt+zu)$
$(yu+yt+vz+vu+zt) \quad +(yv+yz+vt+zu+ut)$

and consequently the expression for the same, which is a function of b c d e, must have an irrational part which has 12 harmonic significations.

Of the above symmetrical halves the sum of any two $= b$, or of any six $= 3b$, as is signified in I. by their juxtaposition and superposition.

The function $c = (yvz+yvu+yvt+yzu+yzt+yut+vzu+vzt+vut+zut)$ is symmetrically divided if I separate it into two parts (of 5 terms each) whose product $= y^3\,v^3\,z^3\,u^3\,t^3$.

There are 12 such symmetrical halves, viz.:

II. $(yvz+yvt+yut+vzu+zut) \quad +(yvu+yzu+yzt+vzt+vut)$
$(yvz+yzu+yut+vzt+vut) \quad +(yvu+yvt+yzt+vzu+zut)$
$(yvt+yvu+yzu+vzt+zut) \quad +(yvz+yzt+yut+vzu+vut)$
$(yvt+yzu+yzt+vzu+vut) \quad +(yvz+yvu+yut+zut+vzt)$
$(yvz+yvu+yzt+vut+zut) \quad +(yvt+yzu+yut+vzu+vzt)$
$(yvu+yzt+yut+vzu+vzt) \quad +(yvt+yvz+yuz+vut+zut)$

of which the sum of any two $= c$ and of any six $= 3c$, as this is signified by their disposition.

There are relations between the symmetrical halves of b and those of c. I mention one of these. Among the halves of b and c there are always two whose terms multiplied singly with one another give the product $y\,v\,z\,u\,t$ five times. Such corresponding halves are:

$m = (yv+yt+vz+zu+ut)$ with $(yvz+yvt+yut+vzu+zut) = p$ and
$m_{,} = (yz+yu+vu+vt+zt)$ with $(yvu+yzu+yzt+vzt+vut) = p$.

Hence $mp_{,} = (yvu+yzu+yzt+vzt+vut)\,(yv+yt+vz+zu+ut)$
$= (y^2vzu \;.\;.\;.\;.\; 20\,t.)+(y^2v^2u+y^2t^2z+v^2z^2t+z^2u^2y+u^2t^2v)$. Analogously, $m_{,}p = (y^2vzu \;.\;.\;.\;.\; 20\,t.)+(y^2v^2z+y^2u^2t+v^2u^2z+v^2t^2y+z^2t^2u)$.

I can therefore represent the symmetrical half p by a function of $b\;c\;d\;e$ and m, as I can likewise represent by a homogeneous function

III. $(y^2v^2u+y^2t^2z+v^2z^2t+z^2u^2y+u^2t^2v)$, and also
$(y^2z^2v+y^2u^2t+v^2u^2z+v^2t^2y+z^2t^2u)$.

§. 12.

Let an equation of the 5th degree be constructed with the 5 unknown quantities yv, yt, vz, zu, ut, (the 5 terms of the symmetrical half m).

If I put: $(yv+yt+vz+zu+ut) = m$,
$(yvt+yvz+yut+vzu+zut) = p$,
$d+(y^2vt+v^2yz+z^2vu+u^2zt+t^2yu) = n$,
$(y^2v^2zt+y^2t^2vu+v^2z^2yu+z^2u^2vt+u^2t^2yz) = q$,

it follows that the equation to be constructed is:

I. $x^5 - mx^4 + nx^3 - qx^2 + epx - e^2 = 0$.

Analogously I construct for the unkown quantities yz, yu, vu, vt, zt the equation:

II. $x^5 - m_{,}x^4 + n_{,}x^3 - q_{,}x^2 + ep_{,}x - e^2 = 0$,

in which $m_{,} = b - m$,
$n_{,} = d + (y^2uz+v^2ut+z^2yt+u^2yv+t^2vz)$,
$q_{,} = (y^2u^2vz+y^2z^2ut+v^2u^2yt+v^2t^2uz+z^2t^2yv)$,
$p_{,} = c - p$.

If I now mnltiply together equations I. and II. I thus get an equation of the 10th degree, whose unknown quantities are the ten combinations of y, v, z, u and t taken 2 at a time. This multiplication gives:

III. (1) $x^{10}-(m+m_{,})x^9+(n+n_{,}+m.m_{,})x^8-(q+q_{,}+nm_{,}+n'm)x^7$
$+[e(p+p_{,}+qm_{,}+q_{,}m+nn_{,})]x^6-[2e^2+e(pm_{,}+p_{,}m+q_{,}n+qn_{,})]x^5$
$+[e^2(m+m_{,})+e(pn_{,}+p_{,}n)+qq_{,}]x^4-[e^2(n+n_{,})+e(p_{,}q+q_{,}p)]x^3$
$+[e^2(q+q_{,}+pp_{,})]x^2-e^3(p+p_{,})x+e^4 = 0$

or (2) $x^{10}-bx^9+Bx^8-Cx^7+Dx^6-Ex^5+Fx^4-Gx^3+Hx^2-e^3cx+e^4=0$.

The coefficients of equation III. can be represented by functions of b, c, d, e. Thus:

IV. $B = - d$, $F = be^2 - 3cde + d^3$,
$C = c^2 - 2bd$, $G = e^2(b^2 - d)$,
$D = ce - d^2$, $H = bde^2$,
$E = - 2bce + bd^2 + 2e^2$.

From III. and IV. there results:

V. (1) $m_{,} = b - m,$
(2) $p_{,} = c - p,$
(3) $-d = (n + n_{,} + m.m_{,}),$
(4) $c^2 - 2bd = (q + q_{,} + nm_{,} + n_{,}m),$
(5) $-d^2 = (qm_{,} + q_{,}m + nn_{,}),$
(6) $-2bce + bd^2 = [e(pm_{,} + p_{,}m) + q_{,}n + qn_{,}],$
(7) $-3cde + d^3 = [e(pn_{,} + p_{,}n) + qq_{,}],$
(8) $e(b^2 - d) = [e(n + n_{,}) + qp_{,} + q_{,}p],$
(9) $bd = (q + q_{,} + pp_{,}).$

From (3), $n_{,} = -(d + n + mm_{,})$. From this and (4), (5), (6), (7) and (8):

VI. (1) $(c^2 - 2bd) = (q + q_{,} + nm_{,} - nm - dm - m^2m_{,}),$
(2) $(-d^2) = (qm_{,} + q_{,}m - nd - n^2 - nm_{,}m),$
(3) $(-2bce + bd^2) = [e(pm_{,} + p_{,}m) + q_{,}n - qd - qn - qm.m_{,}),$
(4) $(-3cde + d^3 = [e(p_{,}n - pd - pn - pm.m_{,}) + qq_{,}],$
(5) $(eb^2 = (pq_{,} + p_{,}q - em.m_{,}).$

From V. (9) $q_{,} = bd - q - pp_{,}$. From this and VI.:

VII. (1) $(c^2 - 3bd) = (-pp_{,} + nm_{,} - nm - dm - m^2m_{,}),$
(2) $(-d^2) = (qm_{,} + bdm - pm - mpp_{,} - nd - n^2 - nm.m_{,}),$
(3) $(-2bce + bd^2) = [e(pm_{,} + p_{,}m) + nbd - 2qn - npp_{,} - qd - qm.m_{,}],$
(4) $(-3cde + d^3) = [e(pn_{,} - pd - pn - pm.m_{,}) + bdq - q^2 - qpp_{,}],$
(5) $(eb^2 = (p_{,}q + bdp - pq - p^2p_{,} - em.m_{,}).$

From VII. (1) and (5): $n = \frac{(c^2 - 3bd + pp_{,} + dm + m^2m_{,})}{m_{,} - m},$

$$q = \frac{(eb^2 + p^2p_{,} - pbd + em.m_{,})}{p_{,} - p}.$$

Since $p_{,} = c - p$ and $m_{,} = b - m$, if I substitute these expressions in equations VII. (2), (3) and (4), three other equations arise which have only the unknown quantities p and m. From those again an equation for $m_{,}$ may be found, but which can be constructed more easily (§. 16). While now I do not discover m from these three equations, but, by retaining this element, find p, this must be capable of being done by means of a final equation of the first degree (for p), for in case m is known there is only *one* p which sustains to it the relation shown by equation I. to exist between them. Consequently it is clear that p is a function of b, c, d, e and m, and thus, when m is known, I know p together with $p_{,}$.

Hereby it is also proved that the expressions §. 11. III. are functions of b, c, d, e and m.

§. 13.

Let I.
$$\begin{aligned}
&(25e-bc)+4(y^3vt+y^3zu+v^3yz+v^3ut+z^3yt+z^3vu+u^3yv+u^3zt\\
&+t^3yu+t^3vz)+6(y^2v^2u+y^2z^2v+y^2u^2t+y^2t^2z+v^2z^2t+v^2u^2z+v^2t^2y\\
&+z^2u^2y+z^2t^2u+u^2t^2v) = S.\\
&(y^4v+v^4z+z^4u+u^4t+t^4y)+2(y^3u^2+v^3t^2+z^3y^2+u^3v^2+t^3z^2)\\
&+4(y^3zt+v^3yu+z^3vt+u^3yz+t^3vu)+6(y^2v^2t+y^2t^2u+v^2z^2y\\
&+z^2u^2v+u^2t^2z)+12(y^2vzu+v^2zut+z^2yut+u^2yvt+t^2yvz) = T.\\
&(y^4z+v^4u+z^4t+u^4y+t^4v)+2(y^3v^2+v^3z^2+z^3u^2+u^3t^2+t^3y^2)\\
&+4(y^3ut+v^3yt+z^3yv+u^3vz+t^3zu)+6(y^2z^2u+y^2u^2v+v^2u^2t\\
&+v^2t^2z+z^2t^2y)+12(y^2vzt+v^2yzu+z^2vut+u^2yzt+t^2yvu) = U.\\
&(y^4u+v^4t+z^4y+u^4v+t^4z)+2(y^3t^2+v^3y^2+z^3v^2+u^3z^2+t^3u^2)\\
&+4(y^3vz+v^3zu+z^3ut+u^3yt+t^3vy)+6(y^2z^2t+y^2u^2z+v^2u^2y\\
&+v^2t^2u+z^2t^2v)+12(y^2vut+v^2yzt+z^2yvu+u^2vzt+t^2yzu) = V.\\
&(y^4t+v^4y+z^4v+u^4z+t^4u)+2(y^3z^2+v^3u^2+z^3t^2+u^3y^2+t^3v^2)\\
&+4(y^3vu+v^3zt+z^3yu+u^3vt+t^3yz)+6(y^2v^2z+y^2t^2v+v^2z^2u\\
&+z^2u^2t+u^2t^2y)+12(y^2zut+v^2yut+z^2yvt+u^2yvz+t^2vzu) = W.
\end{aligned}$$

If now I represent the expressions $0 = (y+v+z+u+t)^5$, $f(1) = (y+v'+z''+u'''+t'''')^5$, etc., by the development of the powers indicated, there results:

$$\begin{array}{lll}
\text{II.} & (y+v+z+u+t)^5 = 0 & = 5(S+T+U+V+W)\\
& (y+v'+z''+u'''+t'''')^5 = F(1) & = 5(S+T'+U'+V'''+W'''')\\
& (y+v''+z''''+u'+t''')^5 = F(2) & = 5(S+T''+U'''+V'+W''')\\
& (y+v'''+z'+u''''+t'')^5 = F(3) & = 5(S+T'''+U'+V''''+W'')\\
& (y+v''''+z'''+u''+t')^5 = F(4) & = 5(S+T''''+U''+V''+W')\\
\cline{2-3}
\text{III.} & F(1)+F(2)+F(3)+F(4) = 25S. &
\end{array}$$

If I multiply the expression:

$(y^2vt^{*}+y^2zu+v^2yz+v^2ut+z^2vu+z^2yt+u^2yv+u^2zt+t^2yu+t^2vz)$

by $0 = (y+v+z+u+t)$, there results:

$$\begin{aligned}
0 &= (y^2vzu \ldots 20\,t.)+(y^3vt+y^3zu+v^3yz+v^3ut+z^3vu+z^3yt+u^3yv\\
&\quad+u^3zt+t^3yu+t^3vz)+(y^2v^2z+y^2u^2z+y^2z^2t+y^2t^2u+v^2z^2u+v^2u^2y\\
&\quad+v^2t^2z+z^2u^2t+z^2t^2v+u^2t^2y+y^2v^2t+y^2u^2v+y^2z^2u+y^2t^2v+v^2z^2y\\
&\quad+v^2u^2t+v^2t^2u+z^2u^2v+z^2t^2y+u^2t^2z)\\
&= -5e+(bc+5e)-(y^2v^2u+y^2t^2z+v^2z^2t+z^2u^2y+u^2t^2v+y^2z^2v+y^2u^2t\\
&\quad+v^2u^2z+z^2t^2u+v^2t^2y)+(y^3vt+y^3zu+v^3yz+v^3ut+z^3vu+z^3yt\\
&\qquad\qquad+u^3yv+u^3zt+t^3yu+t^3vz).
\end{aligned}$$

Whence:

$$\begin{aligned}
\text{III.}\ &(y^3vt+y^3zu+v^3yz+v^3ut+z^3vu+z^3yt+u^3yv+u^3zt+t^3yu+t^3vz)\\
&= -bc+(y^2v^2u+y^2t^2z+v^2z^2t+z^2u^2y+u^2t^2v+y^2z^2v+y^2u^2t+v^2u^2z\\
&\qquad+z^2t^2u+v^2t^2y).
\end{aligned}$$

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*For want of sorts Roman letters are here used.—*Compositor.*

If the result of this equation is introduced into the expression for S (I. preceding), there follows:

$$\text{IV.}\quad S = 5[(5e - bc) + 2(y^2v^2u + y^2t^2z + v^2z^2t + z^2u^2y + u^2t^2v + y^2z^2v + y^2u^2t + v^2u^2z + z^2t^2u + v^2t^2y)],$$

which I write $S = 5s$.

From this and III.:

$$\text{V.}\quad F(1) + F(2) + F(3) + F(4) = 125s,$$

and it is shown, by the relations in §. 11. III. and §. 12. VII., that s is a function of b, c, d, e and m.

§. 14.

From §. 7. VII. I take:

$$\text{I.}\quad (1)\ c = \tfrac{1}{25}[(CE^2 + D^2E)F^2 + (C^2 + D)F],$$

$$(2)\ -125bc + 3125e = (E^5F^4 + D^5F^3 + C^5F^2 + F) + 10(C^2DE^2 + D^2E^2)F^3 + 10(C^2E + CD^2)F^2.$$

But now since (§. 7 and §. 13)

$$\begin{aligned}(E^5F^4 + D^5F^3 + C^5F^2 + F) &= (y + v' + z'' + u''' + t'''')^5 + (y + v'' + z'''' + u' + t''')^5 \\ &\quad + (y + v''' + z' + u'''' + t'')^5 + (y + v'''' + z''' + u'' + t')^5 \\ &= F(1) + F(2) + F(3) + F(4) \\ &= 125s,\end{aligned}$$

I write equation (2) thus:

$$(3)\ 25(25e - bc - s) = 2(C^2DE^2 + D^2E^2)F^3 + 2(C^2E + CD^2)F^2.$$

From §. 10. VI. I know:

$\text{II.}\quad f(1)(4) = \frac{1}{2}[-5b + (2m - b)\sqrt{5}]$, which I designate by q,

$\qquad f(2)(3) = \frac{1}{2}[-5b - (2m - b)\sqrt{5}]$, which I designate by p,

thus: $\qquad f(1)(2)(3)(4) = 5(b^2 - m^2 + mb) = pq.$

The expression $f(1)$ etc. having been equated with $CDEF^2$, I have

$$\text{III.}\quad CDF = q \quad\text{or}\quad C = q \div DF, \qquad EF = p \quad\text{or}\quad E = p \div F.$$

Having substituted both in equation I. there follows:

$$\text{IV.}\quad (1)\ D^4pF^2 + D^3F^2 - 25D^2cF + Dp^2q + q^2) = 0,$$

$$(2)\ D^4p^2F^2 + D^3qF^2 + \tfrac{1}{2}[25(s + bc - 25e)]D^2F + Dp^2q^2 + q^2p = 0.$$

Whence, first, from:

$$\begin{aligned}(D^4p^2F^2 + D^3qF^2 + \tfrac{1}{2}[25(s + bc - 25e)]D^2F + Dp^2q^2 + q^2p) &= 0 \\ -(D^4p^2F^2 + D^3pF^2 - \quad 25 \quad\quad cp \quad\quad D^2F + Dp^3q + q^2p) &= 0,\end{aligned}$$

there results: $(q - p)D^3F^2 + \frac{1}{2}[25(s + bc + 2cp - 25e)]D^2F + Dp^2q(q - p) = 0$, or

$$\text{V.}\quad (1)\ D^2F^2 + 25\left(\frac{s + bc + 2cp - 25e}{2(q - p)}\right)DF + p^2q = 0.$$

I put: $$25\left(\frac{s + bc + 2cp - 25e}{2(q - p)}\right) = 2r,$$

thus $\quad (2)\ D^2F^2 + 2rDF + p^2q = 0, \qquad (3)\ DF = -r + \sqrt{(r^2 - p^2q)}.$

Second, from IV. (1) and (2) there results:

$$
\begin{aligned}
&(D^4p^2F^2+D^3qF^2+\tfrac{1}{2}[25(s+bc-25e)]D^2F+Dp^2q^2+q^2p)=0\\
-&(D^4pqF^2+D^3qF^2-\quad 25 \qquad cq \qquad D^2F+Dp^2q^2+q^3\ )=0\\
=\ &D^4F^2p(p-q)+\tfrac{1}{2}[25(s+bc+2cq-25e)]D^2F+\quad q^2(p-q)\quad =0,
\end{aligned}
$$

VI. (1) or $D^4F^2+25\left(\frac{s+bc+2cq-25e}{2p(p-q)}\right)D^2F+\frac{q^2}{p}=0.$

I put $25\left(\frac{s+bc+2cq-25e}{2p(p-q)}\right)=2w.$

thus (2) $D^4F^2+2w.D^2F+q^2\div p=0$, (3) $D^2F=-w+\surd(w^2-q^2\div p)$.

From this and V:

VII. (1) $D=\frac{D^2F}{DF}=\left(\frac{-w\ +\surd(w^2-q^2\div p)}{-r\ +\surd(r^2-p^2q\ \ )}\right)$

$$=\frac{[-wp+\surd(w^2p^2-q^2p)][-r-\surd(r^2-p^2q)]}{p^3q},$$

(2) $C=\frac{q}{DF}=\frac{q}{[-r+\surd(r^2-p^2q)]}=-\frac{[r+\surd(r^2-p^2q)]}{p^2},$

(3) $CD=\frac{[-wp+\surd(w^2p^2-q^2p)][r+\surd(r^2-p^2q)]^2}{p^5q},$

(4) $F=\frac{q}{CD}=\frac{p^5q^2}{[-wp+\surd(w^2p^2-q^2p)][r+\surd(r^2-p^2q)]^2},$

$$=\frac{-[wp+\surd(w^2p^2-q^2p)][r-\surd(r^2-p^2q)]^2}{q^2},$$

(5) $E=\frac{p}{F}=\frac{-q^2p}{[wp+\surd(w^2p^2-q^2p)][r-\surd(r^2-p^2q)]^2},$

$$=\frac{-[wp-\surd(w^2p^2-q^2p)][r+\surd(r^2-p^2q)]^2}{p^4q^2}.$$

From (4) it now follows immediately, since $F(1)=F$, $F(2)=C^5F^2$, $F(3)=D^5F^3$, $F(4)=E^5F^4$,

$$
\begin{aligned}
\text{VIII.}\quad f(1)&=\sqrt[5]{\left(\frac{-[wp+\surd(w^2p^2-q^2p)][r-\surd(r^2-p^2q)]^2}{q^2}\right)},\\
f(2)&=\sqrt[5]{\left(\frac{-[r+\surd(r^2-p^2q)][wp+\surd(w^2p^2-q^2p)]^2}{p^2}\right)},\\
f(3)&=\sqrt{\left(\frac{-[r-\surd(r^2-p^2q)][wp-\surd(w^2p^2-q^2p)]^2}{p^2}\right)},\\
f(4)&=\sqrt{\left(\frac{-[wp-\surd(w^2p^2-q^2p)][r+\surd(r^2-p^2q)]^2}{q^2}\right)}.
\end{aligned}
$$

So that in the construction of the formula for y, the choice is between:

IX. (1) $y=\frac{1}{5}[f+Cf^2+Df^3+Ef^4]$

and $y=\frac{1}{5}[{}^5\surd F(1)+{}^5\surd F(2)+{}^5\surd F(3)+{}^5F(4).$

For use the first form, by which the mutual fitness of the 4 fifth roots is best represented, commends itself.

(To be concluded in No. 2.)